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VISUAL SP AND COMMUNICATIONS

CHAPTER: QUANTIZATION

Quantization maps any continuous input x into a set of discrete values $\hat{x}$. $\hat{x}$ is a discrete variable with values $\hat{x} = \{\hat{x}_1, \hat{x}_2, ..., \hat{x}_k, ...\}$

Let's define some parameters:

1. Reproduction or reconstruction values $\hat{x}$
2. Quantization interval Δ
3. Decision level t_k

The Quantization interval is $\Delta = [t_k, t_{k+1}]$ for a reproduction level $\hat{x}_k$

Quantization can be seen as a function $Q: x \rightarrow \hat{x}$ or $\hat{x} = Q(x)$

We need a metric of the error introduced by the quantization process. We will call it "overall distortion" and it includes all points.

The Mean Square Error is defined as $MSE = [x - Q(x)]^2$

The Average Distortion is: $D = \int_{-\infty}^{\infty} [x - Q(x)]^2 \, p(x) \, dx$; where *p(x)* is the probability density function of the input throughout the entire input interval (all possible values).

That expression can be written as:

$$D = \sum_{k=-\infty}^{\infty} \left[\int_{t_k}^{t_{k+1}} [x - \hat{x}_k]^2 \, p(x) \, dx \right]$$

If we bound our problem for *k=[1,N]* we have:

$$D = \sum_{k=1}^{N} \left[\int_{t_k}^{t_{k+1}} [x - \hat{x}_k]^2 \, p(x) \, dx \right]$$

Minimum Average Distortion Function

We need to find when *D* becomes minimum. This is a minimization problem. Let's set some rules:
Rule 1: Suppose $\{\hat{x}_k\}$ values are fixed. Now we are going to find the values of $\{t_k\}$ that make the distortion minimum.

$$\frac{\partial D}{\partial t_k} = \frac{\partial}{\partial t_k}\sum_{k=1}^{N}\left[\int_{t_k}^{t_{k+1}}\left[x-\hat{x}_k\right]^2 p(x)\,dx\right]$$ we only have two terms with t_k in them

$$\frac{\partial D}{\partial t_k} = \frac{\partial}{\partial t_k}\int_{t_{k-1}}^{t_k}\left[x-\hat{x}_{k-1}\right]^2 p(x)\,dx + \frac{\partial}{\partial t_k}\int_{t_k}^{t_{k+1}}\left[x-\hat{x}_k\right]^2 p(x)\,dx$$

$$\frac{\partial D}{\partial t_k} = \left[t_k-\hat{x}_{k-1}\right]^2 p(t_k) - \left[t_k-\hat{x}_k\right]^2 p(t_k)$$

To find the minimum we equate this to zero:

$\left[t_k-\hat{x}_{k-1}\right]^2 p(t_k) - \left[t_k-\hat{x}_k\right]^2 p(t_k) = 0$ We see that it is independent of *p(x)*

$$\left[t_k-\hat{x}_{k-1}\right]^2 = \left[t_k-\hat{x}_k\right]^2$$
$$t_k^{\ 2} - 2t_k\hat{x}_{k-1} + \hat{x}^2{}_{k-1} = t_k^{\ 2} - 2t_k\hat{x}_k + \hat{x}^2{}_k$$
$$-2t_k\hat{x}_{k-1} + 2t_k\hat{x}_k = \hat{x}^2{}_k - \hat{x}^2{}_{k-1}$$
$$2t_k\left(-\hat{x}_{k-1}+\hat{x}_k\right) = \hat{x}^2{}_k - \hat{x}^2{}_{k-1}$$
$$2t_k\left(-\hat{x}_{k-1}+\hat{x}_k\right) = \left(-\hat{x}_{k-1}+\hat{x}_k\right)\left(\hat{x}_{k-1}+\hat{x}_k\right)$$
$$\boxed{t_k = \frac{(\hat{x}_{k-1}+\hat{x}_k)}{2}}$$

Therefore the values of $\{t_k\}$ that make the distortion minimum are the central values. This is called the **Middle Point Conditions**.

Rule 2: Suppose $\{t_k\}$ values are fixed. Now we are going to find the values of $\{\hat{x}_k\}$ that make the distortion minimum.

$$\frac{\partial D}{\partial \hat{x}_k} = \frac{\partial}{\partial \hat{x}_k}\sum_{k=1}^{N}\left[\int_{t_k}^{t_{k+1}}\left[x-\hat{x}_k\right]^2 p(x)\,dx\right]$$ we only have one terms with x_k in it

$$\frac{\partial D}{\partial \hat{x}_k} = \frac{\partial}{\partial \hat{x}_k}\left[\int_{t_k}^{t_{k+1}}\left[x-\hat{x}_k\right]^2 p(x)\,dx\right] = \left[\int_{t_k}^{t_{k+1}}\frac{\partial\left[x-\hat{x}_k\right]^2}{\partial \hat{x}_k}\,p(x)\,dx\right] = \left[\int_{t_k}^{t_{k+1}} -2\left[x-\hat{x}_k\right]p(x)\,dx\right]$$

To find the minimum we equate this to zero:

$0 = \int_{t_k}^{t_{k+1}} -2\left[x-\hat{x}_k\right]p(x)\,dx$ $0 = \int_{t_k}^{t_{k+1}} -2x\,p(x)\,dx + \int_{t_k}^{t_{k+1}} 2\hat{x}_k\,p(x)\,dx$

$\int_{t_k}^{t_{k+1}} 2x\,p(x)\,dx = \int_{t_k}^{t_{k+1}} 2\hat{x}_k\,p(x)\,dx$ Finally

$$\boxed{\hat{x}_k = \frac{\int_{t_k}^{t_{k+1}} x\,p(x)\,dx}{\int_{t_k}^{t_{k+1}} p(x)\,dx}}$$

Therefore the values of $\{\hat{x}_k\}$ that make the distortion minimum are the centroid values. This is called the **Centroid Conditions**.

Minimum Rate Function for a General Quantizer.

The objective is to find a function $D = f(R)$ function of the rate *R* and we will call this function "Distortion Rate". If we define distortion as the MSE we have:

$$D = \sum_{k=1}^{N}\left[\int_{t_k}^{t_{k+1}}\left[x-\hat{x}_k\right]^2 p(x)\,dx\right]$$

First, we will obtain the minimum rate:

And the rate is bounded by the Entropy after quantization. Therefore:

$$R_{\min} = H_q(X)$$

Being

$$H_q(X) = -\sum_{k=1}^{N} P_k \log_2\left(P_k\right)$$

Being:

$P_k = \int_{t_k}^{t_{k+1}} p(x)dx$. Plugging in:

$$H_q(X) = -\sum_{k=1}^{N}\left(\int_{t_k}^{t_{k+1}} p(x)dx\right)\log_2\left(\int_{t_k}^{t_{k+1}} p(x)dx\right)$$

First assumption $p(x) = p(k\Delta_k)$ is constant in the interval $t_{k+1} - t_k = \Delta_k$. Therefore:

$$H_q(X) = -\sum_{k=1}^{N}\left(\int_{t_k}^{t_{k+1}} p(k\Delta_k)dx\right)\log_2\left(\int_{t_k}^{t_{k+1}} p(k\Delta_k)dx\right)$$

$$H_q(X) = -\sum_{k=1}^{N}\left(p(k\Delta_k)\Delta_k\right)\log_2\left(p(k\Delta_k)\Delta_k\right) = -\sum_{k=1}^{N}\left(p(k\Delta_k)\Delta_k\right)\left[\log_2 p(k\Delta_k) + \log_2 \Delta_k\right]$$

$$= -\sum_{k=1}^{N} p(k\Delta_k)\Delta_k \log_2 p(k\Delta_k) - \sum_{k=1}^{N} p(k\Delta_k)\Delta_k \log_2 \Delta_k$$

Second assumption: *N* is large; therefore the summation changes to an integral

$$H_q(X) = -\int_a^b p(x)\log_2 p(x)\,dx - \int_a^b p(x)\log_2\left(\Delta_k\right)dx$$

In the case of **<u>Uniform Quantizer</u>** $\Delta_k = \Delta$

$H_{uq}(X) = h(x) - \log_2 \Delta$ where $h(x)$ is the entropy of *x* the input signal.

Therefore the minimum rate for a Uniform Quantizer is

$\boxed{R_{\min,uq} = h(x) - \log_2 \Delta}$ **Minimum Rate for Uniform Quantizer with respect the input entropy and the step size.**

We need now an expression of the Distortion with respect the step size (Δ).

Distortion Rate for Uniform Quantizer

The Uniform Quantizer divides the input range B in N quantization steps of the same size Δ, or $B = N\Delta$. If all N steps are coded with fixed code length, the average bit per sample or bit rate is $R = \log_2 N$ or $2^R = N$. The step size can be written as:

$$\Delta = \frac{B}{N} = B2^{-R}$$

Let's particularize the general Distortion expression:

$$D = \sum_{k=1}^{N} \left[\int_{t_k}^{t_{k+1}} \left[x - \hat{x}_k \right]^2 p(x)\, dx \right]$$

For Uniform Quantizer we have

$$D_{uq} = \sum_{k=1}^{N} \left[\int_{k\Delta}^{k\Delta+\Delta} \left[x - k\Delta - \frac{\Delta}{2} \right]^2 p(k\Delta)\, dx \right] \sum_{k=1}^{N} \left[p(k\Delta) \frac{\left[x - k\Delta - \frac{\Delta}{2} \right]^3}{3} \Bigg|_{k\Delta}^{k\Delta+\Delta} \right]$$

Finally

$$D_{uq} = \sum_{k=1}^{N} \left[p(k\Delta) \frac{\Delta^3}{12} \right] = \frac{\Delta^2}{12} \sum_{k=1}^{N} \left[p(k\Delta)\Delta \right] = \frac{\Delta^2}{12}$$

$\boxed{D_{uq} = \frac{\Delta^2}{12}}$ **Distortion for Uniform Quantizer function of the step size.**

We can find the quantizer interval expression with respect the rate. Previously we found that:

$R_{\min,uq} = h(x) - \log_2 \Delta$ Therefore: $\Delta = 2^{h(x) - R_{\min,uq}}$

Now, we can find a relationship between R and D as:

$\boxed{D_{uq} = \frac{\Delta^2}{12} = \frac{2^{2h(x)}}{12} 2^{-2R_{\min,uq}}}$ **Distortion Rate for UQ (distortion function of the min. rate).**

Observation:
Let's calculate the Variance of the uniform quantizer We assume that the quantizer error is a uniform random variable between its bounds, $[-\frac{\Delta}{2},\frac{\Delta}{2}]$. Then $p(e)=\frac{1}{\Delta}$ and $\mu=0$
Therefore its variance will be:

$$\sigma_{uq}^2 = E[(e-\mu)^2] = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2\, p(e)\, de = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} e^2 \frac{1}{\Delta} de = \frac{1}{\Delta}\frac{e^3}{3}\Bigg|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{1}{3\Delta}\left[(\frac{\Delta}{2})^3 - (\frac{-\Delta}{2})^3\right]$$

$$= \frac{1}{3\Delta}\left[\frac{\Delta^3}{2^2}\right] = \frac{\Delta^2}{12}$$

This is the same value as the distortion D_q if it is defined as the MSE.

$$D_{uq} = \sigma_{uq}^2$$

Let's find the relation between the variance after the quantizer and the variance of the input signal for different types of input PDF's.

Uniform Quantizer with Uniformly Distributed RV Input

The objective is to find the Distortion function (quantizer variance) with respect the input variance.

Imagine the case that the source is uniformly distributed along the range $B = N\Delta$.
Therefore the pdf is $p(x) = \frac{1}{B}$.
The input variance is

$$\sigma^2{}_x = E[x^2] = \int_{-\frac{B}{2}}^{\frac{B}{2}} x^2\, p(x)\, dx = \int_{-\frac{B}{2}}^{\frac{B}{2}} x^2 \frac{1}{B} dx = \frac{1}{B}\frac{x^3}{3}\Bigg|_{-\frac{B}{2}}^{\frac{B}{2}} = \frac{1}{3B}\left(\frac{B^3}{2^2}\right) = \frac{B^2}{12} = \frac{N^2\Delta^2}{12}$$

$$\sigma^2{}_x = \frac{N^2\Delta^2}{12}$$

We found that for a Uniform quantizer, the variance is:

$$D_{uq} = \sigma^2{}_{uq} = \frac{\Delta^2}{12} = \frac{2^{2h(x)}}{12} 2^{-2R_{\min,uq}}$$

Therefore:
$\sigma^2{}_x = N^2\sigma^2{}_{uq}$ or

$\boxed{D_{uq} = \sigma^2{}_{uq} = N^{-2}\sigma^2{}_x}$ **Variance (DR) of the UQ with U distributed input vs. the input variance.**

For the case of fixed code length (which is the best choice if the input is uniformly distributed), we have $2^R = N$ Therefore we have our final equation:
<u>Distortion function (quantizer variance) with respect the input variance</u>

$\boxed{D_{uq} = \sigma_x 2^{-2R_{\min,uq}}}$ **Distortion Rate for a UQ with uniformly distributed input**

Observation:
If we compare this equation with
$\sigma^2{}_{uq} = \dfrac{2^{2h(x)}}{12} 2^{-2R_{uq}}$ we end up with

$\sigma_x = \dfrac{2^{2h(x)}}{12}$ and because $\sigma_x = \dfrac{B^2}{12}$ we end up with

$B = 2^{h(x)}$ or $h(x) = \log_2 B$

This is logical, since the input signal is a uniform RV in the range *B*. Its entropy is defined as:

$$h(x) = -\int_{\frac{-B}{2}}^{\frac{B}{2}} p(x)\log_2(p(x))\,dx = -\int_{\frac{-B}{2}}^{\frac{B}{2}} \frac{1}{B}\log_2(\frac{1}{B})\,dx = -\log_2(\frac{1}{B}) = \log_2 B$$

Minimum Rate for a Uniform Quantizer with Uniform Distributed RV Input

$R_{uq\min} = H_{uq}(X) = h(x) - \log_2 \Delta$
Because $h(x) = \log_2 B$
$R_{uq\min} = H_{uq}(X) = \log_2 B - \log_2 \Delta = \log_2 \dfrac{B}{\Delta} = \log_2 N$

$\boxed{R_{uq\min} = H_{uq}(X) = \log_2 N}$
$\boxed{R_{uq\min} = H_{uq}(X) = \log_2 B - \log_2 \Delta}$

Another way to find this out is from the distortion rate formula:

$\sigma^2{}_{uq} = \sigma_x 2^{-2R_{\min,uq}}$ Therefore

$R_{\min,uq} = \dfrac{1}{2}\log_2 \dfrac{\sigma_x}{\sigma^2{}_{uq}}$ and we know that $\sigma^2{}_{uq} = \dfrac{1}{N^2}\sigma^2{}_x$. Therefore

$$R_{\min,uq} = \frac{1}{2}\log_2 N^2 = \log_2 N$$

Uniform Quantizer with Gaussian Distributed RV Input

If the input is Gaussian we have pdf is $p(x) = \frac{1}{\sqrt{\sigma_x^2 2\pi}} e^{\frac{-x^2}{2\sigma_x^2}}$ (assuming zero mean).

It can be proved that the entropy is

$$h(x) = \frac{1}{2}\log(2\pi e \sigma_x^2)$$

Therefore applied to our **Uniform quantizer** we have:

$$D_{uq} = \frac{2^{2h(x)}}{12} 2^{-2R_{uq}} = \frac{2^{2\left(\frac{1}{2}\log(2\pi e\sigma_x^2)\right)}}{12} 2^{-2R_{uq}} = \frac{\pi e \sigma_x^2}{6} 2^{-2R_{uq}} = \varepsilon^2 \sigma_x^2 2^{-2R_{uq}} \text{ where } \varepsilon^2 = \frac{\pi e}{6}$$

$\boxed{\sigma^2{}_{uq} = \varepsilon^2 \sigma_x^2 2^{-2R_{uq}}}$ **Distortion Rate for UQ with Gaussian Input Distribution.**

The entropy of this quantizer will be:
Average bit rate for a Uniform Quantizer with Gaussian Distributed RV Input

$$R_{uq\min} = H_{uq}(X) = h(x) - \log_2 \Delta$$

In this case we have:

$$h(x) = \frac{1}{2}\log_2(2\pi e \sigma_x^2)$$

Therefore:

$$R_{uq\min} = H_{uq}(X) = \frac{1}{2}\log_2(2\pi e\sigma_x^2) - \log_2 \Delta$$

$$R_{uq\min} = H_{uq}(X) = \frac{1}{2}\log_2(2\pi e\sigma_x^2) - \log_2 \frac{B}{N}$$

$$R_{uq\min} = H_{uq}(X) = \frac{1}{2}\log_2(2\pi e\sigma_x^2) + \log_2 \frac{N}{B}$$

$$R_{uq\min} = H_{uq}(X) = \frac{1}{2}\log_2(2\pi e\sigma_x^2) - \frac{1}{2}\log_2 B^2 + \log_2 N$$

$$R_{uq\min} = H_{uq}(X) = \frac{1}{2}\log_2(12\varepsilon\sigma_x^2) - \frac{1}{2}\log_2(B^2) + \log_2 N \text{ where } \varepsilon^2 = \frac{\pi e}{6}$$

$$R_{uq\min} = H_{uq}(X) = \frac{1}{2}\log_2 \varepsilon + \frac{1}{2}\log_2(\sigma_{xGauss}^2) - \frac{1}{2}\log_2(\frac{B^2}{12}) + \log_2 N$$

Uniform Quantizer with Deterministic Input
If the input signal is deterministic then, the expression of the quantizer entropy becomes:

$H_q(X) = R_q = h(x) - \log_2 \Delta$ with $h(x) = 0$ then:
$H_q(X) = R_q = -\log_2 \Delta$

Other way to find this result is:

The distortion Expression for the Uniform Quantizer is:
$\sigma^2{}_{uq} = \frac{2^{2h(x)}}{12} 2^{-2R_{uq}}$ with $h(x) = 0$ Therefore

$\boxed{\sigma^2{}_{uq} = \frac{1}{12} 2^{-2R_{uq}}}$ **Distortion Rate function for UQ with Deterministic Input**

Because we had:

$\sigma^2{}_{uq} = \frac{\Delta^2}{12}$ We can equate both expressions to get:
$\frac{\Delta^2}{12} = \frac{1}{12} 2^{-2R_{uq}}$

Therefore:

$\Delta^2 = 2^{-2R_{uq}}$ or $\Delta = 2^{-R_{uq}}$
$\boxed{R_{uq} = -\log_2 \Delta}$ **Minimum Rate for UQ with Deterministic input.**

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ENTROPY CONSTRAINTED QUANTIZER DESIGN

The previous study about minimizing the Average Distortion of the quantizer was based only on the constraint of minimizing the Distortion.
If we want to minimize the Average Distortion with the Entropy constraint we will find maybe other solutions.

$$D = \sum_{k=1}^{N} \left[\int_{t_k}^{t_{k+1}} \left[x - \hat{x}_k \right]^2 p(x)\, dx \right]$$

The entropy constraint is:

$H_q(X) = R_{\min} \le R_0$ meaning that the rate is larger than the entropy

We can use the Lagrange Optimization Product. However we will use a heuristic approach to see if this problem has solution:

Suppose that R_0 is very small. Therefore the Entropy is small. At the limit, the entropy is zero, which means that all points are located at one quantizer level $\hat{x}_k = \hat{x}_j$

Suppose that R_0 is big. The minimum R_0 value will be given by the maximum value of the entropy. This happens when all the symbols have uniform distribution.
$H_q(X) = \log_2 N$.

We can use Lloyd-Max algorithm to find the solution. This algorithm places step sizes smaller when the pdf is larger (more quantization levels for more probable input values).

Let's compare both quantizers.
First quantizer is Lloyd-Max, with average distortion $D_{Lloyd-Max}$.
Second quantizer is an uniform quantizer with D_{uq}

The Lloyd-Max quantizer has less distortion, because was optimized. Therefore

$$D_{Lloyd-Max} \le D_{uq}$$

However because entropy constraint, what is the best?

Remember that:

$$H_q(X) = -\sum_{k=1}^{N} P_k \log_2 (P_k)$$

For Uniform Quantizer, we have different values for P_k. However, for Lloyd-Max optimizer quantizer we will have more similar values for P_k. Just note that

$P_k = \int_{t_k}^{t_{k+1}} p(x)dx$ and because the centroid condition $\hat{x}_k = \dfrac{\int_{t_k}^{t_{k+1}} x\, p(x)\, dx}{\int_{t_k}^{t_{k+1}} p(x)\, dx}$ the quantization points will be closer as the pdf is higher.

Therefore when the pdf *p(x)* is larger the interval $\Delta_k = [t_k, t_{k+1}]$ is smaller and hence, the P_k tends to be constant for all intervals.

Example: Let be a optimized quantizer where $P_k = \{0.25, 0.25, 0.25, 0.25\}$ and an uniform quantizer where $P_k = \{0.1, 0.4, 0.4, 0.1\}$
The entropies will be:

-0.25*log2(0.25)-0.25*log2(0.25)-0.25*log2(0.25)-0.25*log2(0.25)= 2

-0.1*log2(0.1)-0.4*log2(0.4)-0.4*log2(0.4)-0.1*log2(0.1)= 1.7219

Therefore, surprisingly the optimized quantizer has larger entropy.
In general:
$H_{uq}(X) \le H_{Lloyd-Max}(X)$ therefore
$R_{uq}(X) \le R_{Lloyd-Max}(X)$

Under Entropy constraints, the Uniform Quantizer is better.

EOP

APPENDIX

Gaussian probability function

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Lloyd-Max algorithm. Matlab realization.

Interval: [a, b]. Number of quantization steps N.

Step 1: fix x_k and find t_k by the middle point formula
At the first round, I set up the x_k points uniformly along the interval [a, b].

$$x_k = a + \Delta(k-1) + \frac{\Delta}{2} \text{ for } k = [1, N-1]$$

I find $t_k = \frac{x_k + x_{k-1}}{2}$ with $t_1 = a$ and $t_{N+1} = b$

Step 2: fix t_k and find x_k by the centroid formula

$$\hat{x}_k = \frac{\int_{t_k}^{t_{k+1}} xp(x)\,dx}{\int_{t_k}^{t_{k+1}} p(x)\,dx} \text{ if we consider } p(x) = p(x_k) \text{ constant}$$

$$x_k = \frac{\int_{t_k}^{t_{k+1}} xp(x_k)\,dx}{\int_{t_k}^{t_{k+1}} p(x_k)\,dx} = \frac{p(x_k)\int_{t_k}^{t_{k+1}} x\,dx}{p(x_k)\int_{t_k}^{t_{k+1}} dx} = \frac{\int_{t_k}^{t_{k+1}} x\,dx}{\int_{t_k}^{t_{k+1}} dx} = \frac{\left.\frac{x^2}{2}\right|_{t_k}^{t_{k+1}}}{\left.x\right|_{t_k}^{t_{k+1}}} = \frac{1}{2}\frac{t_{k+1}{}^2 - t_k{}^2}{t_{k+1} - t_k} = \frac{t_{k+1} + t_k}{2}$$

Of course this is the case in Uniform distribution, where $p(x) = p(x_k)$ is constant. The quantization points x_k do not change at all.

For Matlab purposes, we have to discretize the x points. Therefore our formula changes to:

$$\hat{x}_k = \frac{\sum_{i=t_k}^{t_{k+1}} x_i p(x_i)}{\sum_{i=t_k}^{t_{k+1}} p(x_i)}$$

Imagine we have 16 equidistant samples. And 4 levels of Quantization. The interval is from 0 to 4. Therefore the samples are located at:
x = [0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, 2.25, 2.5, 2.75, 3, 3.25, 3.5, 3.75, 4]

the starting quantization levels are
xk = [0.5000, 1.5000, 2.5000, 3.5000]

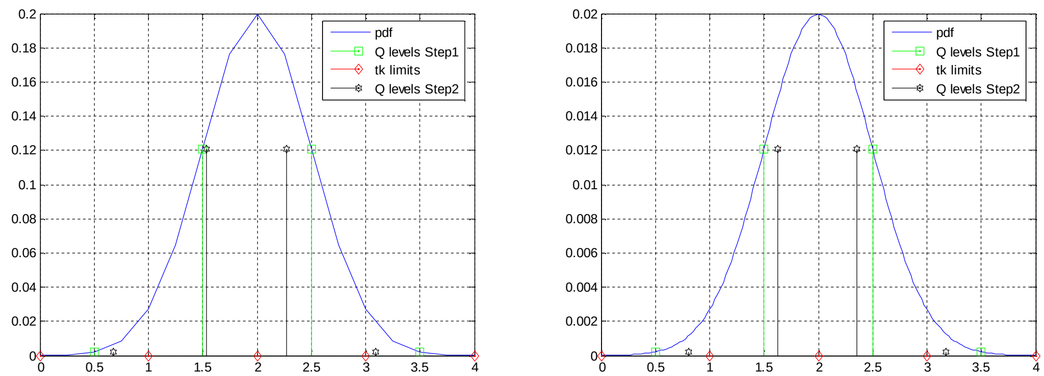
The starting boundary levels are
tk=[0, 1, 2, 3, 4]

Imagine a Gaussian distribution. Let me write the probabilities of each sample.
pk=[0.0001, 0.0004, 0.0022, 0.0088, 0.027, 0.0648, 0.121, 0.176, 0.1995, 0.1760, 0.1210, 0.0648, 0.0270, 0.0088, 0.0022, 0.0004, 0.0001] Of course the sum of all probabilities is 1.

When calculating the next quantization levels:

$$\hat{x}_k = \frac{\sum_{i=t_k}^{t_{k+1}} x_i p(x_i)}{\sum_{i=t_k}^{t_{k+1}} p(x_i)}$$

We find a problem. The samples weight a lot the result of the numerator. Hence, the new quantization levels are not symmetric with respect the Gaussian distribution, although we know that that should be the case (see figure to the left)



This can be fixed if we increase the number of samples. For example if we analyze the case for 40 samples per quantization step (160 total samples) we see that the new quantization levels move symmetrically with respect the Gaussian distribution. In this last case the Distortion decreases from 0.0842 to 0.0612

The code so far, calculates just one step of the Lloyd-Max algorithm. I have to make it recurrent until the Distortion stop decreasing by a set quantity.

Matlab code:

```
%mquantizer01
%Luis M 2/14/2005
%This file draws a quantizer with gaussian pdf

clc
clear all
close all
disp(['START MQUANTIZER01 ///////////////////////////////// ']);
%Quantizer limits
a=0;
b=4;
%Quantizer steps
N =4;
```

```
disp(['Q steps:',num2str(N)]);

delta = (b-a)/N;
disp(['delta (Q step size): ',num2str(delta)]);

%pdf mean and variance
mu = (a+b)/2;
sigma = .5;
%xresolution to set the samples. This number should be high for the
%algorithm to work like in continuous time samples.
div = 20;
xres = delta/(div*2);

%xvector
x = a:xres:b;
lx = length(x);
pdfx = 1/(sigma*sqrt(2*pi))*exp(-(x-mu).^2/(2*sigma^2));
% pdfx = normpdf(x,mu,sigma);
sumprob = sum(pdfx);
%Normalize the pdf so the integral is one
pdfx = pdfx/sumprob;
disp(['Sum of probabilities: ',num2str(sum(pdfx))]);

xperdelta = round(length(x)/N);
disp(['samples per delta: ',num2str(xperdelta)]);

%LLOYD-MAX algorithm.
%Step 1: Fix xk and find tk (middle point law)
%Choosing a uniform distribution for xk
for i=1:N
    xk(i,1)=a+delta*(i-1)+delta/2;
    %This is just to drawing purposes, set the step amplitude as high as
    %the pdf.
    xkx(i)=pdfx(div*i+1+(i-1)*div);
end

%Choosing middle points for tk
tk(1)=a;
tk(N+1)=b;
for k = 2:N
    tk(k)=(xk(k,1)+xk(k-1,1))/2;
end

%Find the Distortion or Average Quantization Error.
%k runs for each quantization step
for k=1:N
    %Accumulate distortion
    accd = 0;
    %index to find each x data inside a quantization step
    i = 1+(k-1)*xperdelta;
    %j runs for each data sample inside a quant. step
    for j=tk(k):xres:tk(k+1)-xres
        accd=accd+(x(i)-xk(k,1))^2*pdfx(i);
        i=i+1;
    end
    dis(k,1)=accd;
end

%Step 2: Fix tk and find the new xk (centroid law)
%k runs for each quantization step
for k=1:N
    %Calculate numerator
    num = 0;
    den = 0;

    %index to find each x data inside a quantization step
    i = 1+(k-1)*xperdelta;
    %disp(['        k:      ',num2str(k)]);

    %j runs for each data sample inside a quant. step
    for j=tk(k):xres:tk(k+1)-xres
```

```
        %disp(['j: ',num2str(j),' x: ',num2str(x(i))]);
        %disp(['i:      ',num2str(i)]);
        %disp(['x: ',num2str(x(i)),' px: ',num2str(pdfx(i))]);
        num=num+x(i)*pdfx(i);
        den=den+pdfx(i);
        i=i+1;
    end
    xk(k,2)=num/den;
end

xk

%Find the Distortion or Average Quantization Error.
%k runs for each quantization step
for k=1:N
    %Accumulate distortion
    accd = 0;
    %index to find each x data inside a quantization step
    i = 1+(k-1)*xperdelta;
    %j runs for each data sample inside a quant. step
    for j=tk(k):xres:tk(k+1)-xres
        accd=accd+(x(i)-xk(k,2))^2*pdfx(i);
        i=i+1;
    end
    dis(k,2)=accd;
end
dis
sumdist = sum(dis)

%left to do is to make this algorithm recursive until the distortion stop
%decreasing by a set quantity.


%Plot figures
%Plot the probability density function of the data
plot(x,pdfx), grid on
hold on;

stem(xk(:,1),xkx,'gs');

tkx = 0*ones(1,length(tk));
stem(tk,tkx,'rd');

stem(xk(:,2),xkx,'kh'); zoom on
legend('pdf','Q levels Step1','tk limits','Q levels Step2');

disp(['END MQUANTIZER01 //////////////////////////////// ']);
```

Screen shot:

```
START MQUANTIZER01 ////////////////////////////////
Q steps:4
delta (Q step size): 1
Sum of probabilities: 1
samples per delta: 40

xk =

   0.5000   0.8033
   1.5000   1.6292
   2.5000   2.3521
   3.5000   3.1744


dis =

   0.0025   0.0006
   0.0370   0.0292
   0.0414   0.0307
   0.0032   0.0007
```

```
sumdist =

    0.0842    0.0612

END MQUANTIZER01 ///////////////////////////////
```

EOD